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# Planar graphs without adjacent cycles of length at most seven are 3-colorable

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ABSTRACT

We prove that every planar graph in which no $i$-cycle is adjacent to a $j$-cycle whenever $3 \leq i \leq j \leq 7$ is 3-colorable and pose some related problems on the 3-colorability of planar graphs.



## 1. Introduction

In 1976, Appel and Haken proved that every planar graph is 4-colorable [3,4], and as early as 1959, Grötzsch [13] proved that every planar graph without 3-cycles is 3-colorable. As proved by Garey, Johnson and Stockmeyer [12], the problem of deciding whether a planar graph is 3-colorable is NP-complete. Therefore, some sufficient conditions for planar graphs to be 3-colorable were stated. In 1976, Steinberg [19] raised the following:

*Steinberg's conjecture '76. Every planar graph without* 4- *and* 5-*cycles is* 3-*colorable.*

In 1969, Havel [14] posed the following problem:

*Havel's problem '69. Does there exist a constant $C$ such that every planar graph with the minimum distance between triangles at least $C$ is* 3-*colorable?*

Havel [15,16] proved that if $C$ exists, then $C \geq 2$, which was improved to $C \geq 4$ by Aksionov and Mel'nikov [2] and, independently, by Steinberg (see [2]).

These two challenging problems remain open. In 1991, Erdős suggested the following *relaxation of Steinberg's Conjecture*: Determine the smallest value of $k$, if it exists, such that every planar graph without cycles of length from 4 to $k$ is 3-colorable. Abbott and Zhou [1] proved that such a $k$ does exist, with $k \leq 11$. This result was later on improved to $k \leq 10$ by Borodin [5] and to $k \leq 9$ by Borodin [6] and Sanders and Zhao [18]. The best known bound for such a $k$ is 7, and it was proved by Borodin, Glebov, Raspaud, and Salavatipour [10]:

**Theorem 1** (*[10]*). *Every planar graph without cycles of length from* 4 *to* 7 *is* 3-*colorable.*

At the crossroad of Havel's and Steinberg's problems, Borodin and Raspaud [11] proved that every planar graph without 3-cycles at distance less than four and without 5-cycles is 3-colorable. (The distance here was improved to three by Borodin and Glebov [7] and Xu [20], and recently it was decreased to two by Borodin and Glebov [8].) Furthermore, Borodin and Raspaud [11] proposed the following conjecture:

*Strong Bordeaux conjecture '03. Every planar graph without* 5-*cycles and without adjacent triangles is* 3 *colorable.*

By adjacent cycles we mean those with at least one edge in common.

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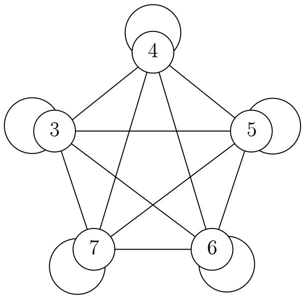


**Fig. 1.** The non-adjacency graph $G_{(A)}$.

Obviously, this conjecture implies *Steinberg's Conjecture*. In [9], Borodin, Glebov, Jensen and Raspaud considered the adjacency between cycles in planar graphs, where all lengths of cycles are authorized; in a sense, this kind of problems is related to *Havel's Problem*. More specifically, they proved that every planar graph without triangles adjacent to cycles of length from 3 to 9 is 3-colorable and proposed the following conjecture:

*Novosibirsk 3-color conjecture '06. Every planar graph without* 3*-cycles adjacent to cycles of length* 3 *or* 5 *is* 3*-colorable.*

Clearly, this one implies both the *Strong Bordeaux Conjecture* and *Steinberg's Conjecture*.

Many other sufficient conditions for the 3-colorability of planar graphs were proposed in which cycles with lengths from specific sets are forbidden (for example, see [21]). In this note we consider an approach based on the adjacencies of cycles. Let us start with some definitions:

*GA — Graph of non-adjacencies*. A *graph of non-adjacencies* is one whose vertices are labelled by integers greater than two and each integer appears at most once. Given a graph $H$ of non-adjacencies, we say that a graph $G$ *respects* $H$ if no two cycles of lengths $i$ and $j$ are adjacent in $G$ if the vertices labelled with $i$ and $j$ are adjacent in $H$.

**Example.** Let $G_{(A)}$ be the graph depicted by Fig. 1. A graph $G$ respecting $G_{(A)}$ is a graph in which there is no $i$-cycle adjacent to a $j$-cycle for $3 \le i \le j \le 7$.

We propose the following natural general question:

**Problem 1.** Under which conditions of adjacencies is a planar graph 3-colorable?

Our main result in this note (proved in Section 2) is that each planar graph respecting the graph $G_{(A)}$ depicted by Fig. 1 is 3-colorable:

**Theorem 2.** *Every planar graph in which no i-cycle is adjacent to a j-cycle whenever* $3 \le i \le j \le 7$ *is* 3*-colorable.*

Clearly, Theorem 2 is an extension of the above mentioned result by Borodin, Glebov, Raspaud, and Salavatipour [10].

The model of non-adjacencies can be made more precise. Define a function $f$ on the edges of a non-adjacency graph $H$ by putting:

- $f(ij) = -1$ if the cycles of lengths $i$ and $j$ are not adjacent in $G$,
- $f(ij) = 0$ if the cycles of lengths $i$ and $j$ are not intersecting in $G$,
- $f(ij) = k$ if the distance between cycles of lengths $i$ and $j$ in $G$ is greater than $k$ (the distance between two cycles $C_1$ and $C_2$ is defined as the length of a shortest path between two vertices of $C_1$ and $C_2$).

Montassier, Raspaud, Wang and Wang [17] suggested a *relaxation of Havel's Problem* and proved:

**Theorem 3** (*[17]*).

(1) *Every planar graph in which the cycles of length* 3, 4, 5*, and* 6 *are at distance at least* 3 *from each other is* 3*-colorable.*
(2) *Every planar graph in which the cycles of length* 3, 4*, and* 5 *are at distance at least* 4 *from each other is* 3*-colorable.*

Note that the graphs studied in Theorem 3 respect the graphs of non-adjacencies $G_{(B)}$ and $G_{(C)}$ depicted by Fig. 2.

We conclude this introduction with two specific problems; see Fig. 3.

**Problem 2.** Let $G$ be a planar graph respecting $G_{(D)}$ depicted by Fig. 3. Let $f_0$ be an $i$-face with $3 \le i \le 11$. Prove that every proper 3-coloring of $G[V(f_0)]$ can be extended to the whole graph. ($G[V(f_0)]$ denotes the subgraph of $G$ induced by the vertices of $f_0$.)

**Problem 3.** Let $G$ be a planar graph respecting $G_{(E)}$ (respectively $G_{(F)}, G_{(I)}, G_{(J)}$) depicted by Fig. 3. Prove that $G$ is 3-colorable.

The result on planar graphs respecting $G_{(F)}$ would imply *Steinberg's Conjecture*. The problem on planar graphs respecting $G_{(I)}$ is the *Novosibirsk 3-Color Conjecture*. Finally, the problem on planar graphs respecting $G_{(J)}$ for any finite $k$ would provide the answer to *Havel's Problem*. The first attempt could be to study planar graphs respecting $G_{(E)}$ or subgraphs of $G_{(A)}$ in Fig. 1.

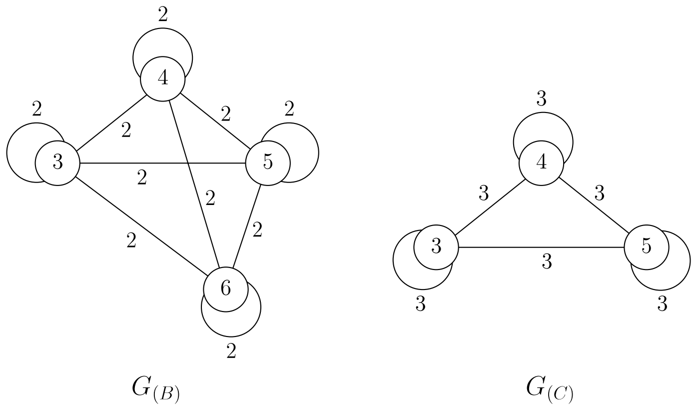


**Fig. 2.** The non-adjacency graphs $G_{(B)}$ and $G_{(C)}$.

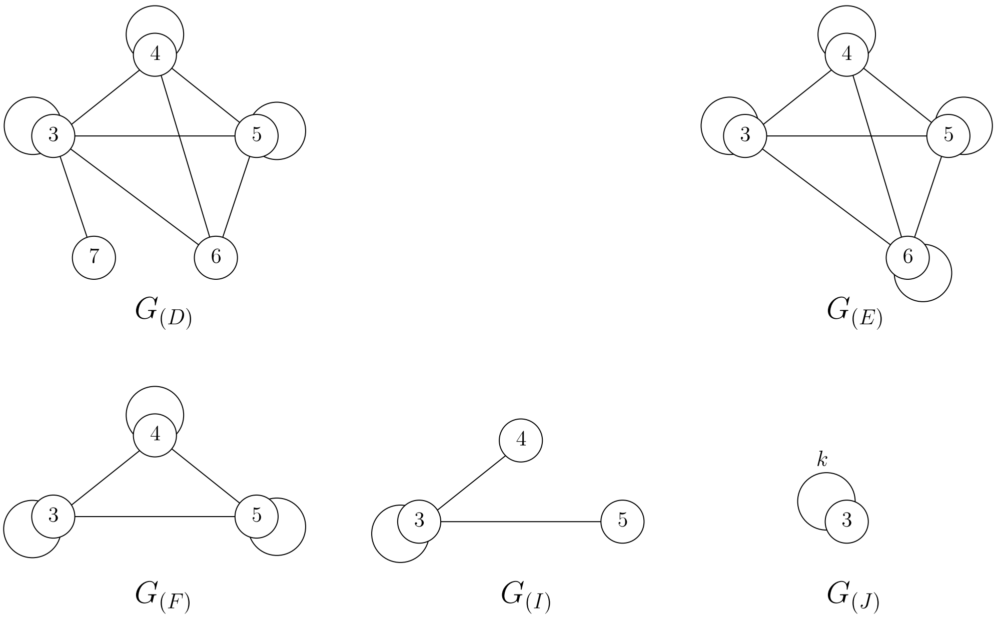


**Fig. 3.** Some non-adjacency graphs.

## 2. Proof of Theorem 2

We first present some notations:

**Notations.** Let $G = (V(G), E(G), F(G))$ be a plane graph, where $V(G)$, $E(G)$ and $F(G)$ denote the sets of vertices, edges and faces of $G$, respectively. The neighbor set and the degree of a vertex $v$ are denoted by $N(v)$ and $d(v)$, respectively. Let $f$ be a face of $G$. We use $b(f)$, $V(f)$ to denote the boundary of $f$, the set of vertices on $b(f)$, respectively. A $k$-vertex (respectively $^{\geq}k$-vertex, $^{\leq}k$-vertex) is a vertex of degree $k$ (respectively at least $k$, at most $k$). The same notation is used for faces and cycles: A $k$-face (respectively $^{\geq}k$-face, $^{\leq}k$-face) is a face of length $k$ (respectively at least $k$, at most $k$). Let $C$ be a cycle of G. By $int(C)$ and $ext(C)$ we denote the sets of vertices located inside and outside $C$, respectively. The cycle $C$ is *a separating cycle* if both $int(C) \neq \emptyset$ and $ext(C) \neq \emptyset$. Let $C$ be a cycle of $G$, and let $u$ and $v$ be two vertices on $C$. We use $C[u, v]$ to denote the path of $C$ clockwisely from $u$ to $v$, and let $C(u, v) = C[u, v] \setminus \{u, v\}$.

Our proof is based on the following coloring extension lemma:

**Lemma 1.** *Suppose $G$ is a connected planar graph respecting $G_{(A)}$ depicted by Fig. 1 and $f_0$ is an $i$-face with $3 \leq i \leq 11$; then every proper 3-coloring of $G[V(f_0)]$ (the subgraph of $G$ induced by the vertices of $f_0$) can be extended to the whole $G$.*

It is easy to see that Lemma 1 implies Theorem 2. Indeed, let $G$ be a minimal counterexample to Theorem 2; clearly, $G$ is connected. If $G$ contains a triangle $C_3$, we assign colors to the vertices of $C_3$ and apply Lemma 1 to $G \setminus int(C_3)$ and to $G \setminus ext(C_3)$. If $G$ does not contain triangles, then $G$ is 3-colorable by Grötzsch's Theorem [13].

So, it suffices to prove Lemma 1. Note that our proof of Lemma 1 is built on the following result by Borodin, Glebov, Raspaud, and Salavatipour [10]:

**Theorem 4.** *Every proper* 3*-coloring of the vertices of any face of length* 8 *to* 11 *in a connected planar graph without cycles of length* 4 *to* 7 *can be extended to a proper* 3*-coloring of the whole graph.*

By $\mathcal{G}$ denote the set of plane graphs that respects $G_{(A)}$ depicted in Fig. 1.
Assume that $G$ is a counterexample to Lemma 1 with:

(1) $c(G) = c_4(G) + c_5(G) + c_6(G) + c_7(G)$ as small as possible, and
(2) $\sigma(G) = |V(G)| + |E(G)|$ minimum under the previous condition

where $c_i(G)$ denotes the number of cycles of length $i$ in $G$.

Without loss of generality, assume that the unbounded face $f_0$ is an $i$-face with $3 \leq i \leq 11$ such that a 3-coloring $\phi$ of $G[V(f_0)]$ cannot be extended to $G$. Let $C_0 = b(f_0)$. All faces different from $f_0$ are called *internal*.

**Claim 1.** *$G$ is* 2*-connected; hence, the boundary of every face is a cycle.*

**Proof.** Observe first that, by the minimality of $G$, there is no cut-vertex in $V(f_0)$. Now assume that $B$ is a pendant block with the cut-vertex $v \in V(G) \setminus V(f_0)$. We first extend $\phi$ to $G \setminus (B \setminus v)$, then we color $B$ with 3 colors by the minimality of $G$ or Grötzsch's Theorem, permute the colors if necessary, and finally get an extension of $\phi$ to $G$. □

**Claim 2.** $\forall v \in int(C_0), d(v) \geq 3$.

**Proof.** Let $v$ be an $^{\leq}2$-vertex with $v \in int(C_0)$. We can first extend $\phi$ to $G \setminus v$ and then color $v$. □

**Claim 3.** *$G$ contains no separating $k$-cycles with* $3 \leq k \leq 11$.

**Proof.** Let $C$ be a separating cycle of length from 3 to 11. By the minimality of $G$, we can extend $\phi$ to $G \setminus int(C)$. Then we extend the 3-coloring of $G[V(C)]$ to $G \setminus out(C)$ using the minimality of $G$. □

**Claim 4.** *$G[V(f_0)]$ is a chordless cycle.*

**Proof.** Let $uv$ be a chord of $C_0$. Then by the minimality of $G$, we can extend $\phi$ to $G \setminus uv$ and so to $G$. □

**Claim 5.** $|f_0| \neq 4, 5, 6, 7$.

**Proof.** Let $C_0 = x_1x_2 \dots x_k$ with $4 \leq k \leq 7$. Let $G'$ be the graph obtained from $G$ by adding $8 - k$ vertices of degree two on the edge $x_1x_2$. Then observe that $c(G') < c(G)$ and $G' \in \mathcal{G}$. By choosing some good colors to the added vertices, we can extend the coloring of the outer face of $G'$ to the whole graph $G'$ by the minimality of $G$. This yields a proper 3-coloring of $G$, a contradiction. □

Now we show that $G$ contains no internal $k$-faces with $4 \leq k \leq 7$. Due to Claim 3 and the cycles' adjacency conditions, every $k$-cycle with $4 \leq k \leq 7$ bounds a face. This will show that $G$ contains no $k$-cycles with $4 \leq k \leq 7$. Finally, Theorem 1 or Theorem 4 will complete the proof of Lemma 1.

**Claim 6.** *$G$ contains no internal* 4*-faces.*

**Proof.** Assume that $G$ contains an internal 4-face $f = x_1x_2x_3x_4$ (the $x_i$'s appear clockwisely on $f$) and $C_f = b(f)$.

Let $G'$ be the graph obtained from $G$ by identifying $x_1$ and $x_3$. Let $x'$ be the obtained vertex. Firstly, we will show that $G' \in \mathcal{G}$ and $c(G') < c(G)$. Secondly, we will show that the identification does not damage the precoloring of $G'$ induced by the precoloring of $G[V(f_0)]$ in $G$. Hence by minimality of $G$, the precoloring of $G'$ will be extendable to whole $G'$ and so it will be for the precoloring of $G[V(f_0)]$ in $G$, a contradiction (since a 3-coloring of $G'$ gives a 3-coloring of $G$ by assigning the color of $x'$ to $x_1$ and to $x_3$).

We show now that $G' \in \mathcal{G}$.

**Observation 1.** (1) *Let $u$ and $v$ be two adjacent vertices on $f$. Let $P_{uv}$ be a shortest path between $u$ and $v$ in $G \setminus \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$ (if any). By the cycles' adjacency conditions, the path $P_{u,v}$ has at least* 8 *vertices.*
(2) *Let $u$ and $v$ be two non-adjacent vertices on $f$. Let $P_{uv}$ be a shortest path between $u$ and $v$ in $G \setminus \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$ (if any). By Claim 3, the path $P_{u,v}$ has at least* 11 *vertices.*

We first show that the identification does not create a $^{\leq}7$-cycle. Suppose to the contrary that $C^*$ is a $^{\leq}7$-cycle in $G'$ created by the identification. The cycle $C^*$ must go through at least two vertices of $x_1, \dots, x_4$ (otherwise, its length cannot decrease by the identification). If $C^*$ goes through two adjacent vertices $u$ and $v$ on $f$, then $C^*$ is composed of a path $P_{u,v}$ of length at least 8, contradicting the size of $C^*$ (by Observation 1.1). If $C^*$ goes through two non-adjacent vertices $u$ and $v$ on $f$, then $C^*$ is composed of a path $P_{u,v}$ of length at least 10, contradicting the size of $C^*$ (by Observation 1.2). Hence the identification does not create $^{\leq}7$-cycles. Moreover the identification does not create an adjacency between two $^{\leq}7$-cycles of $G$, since by hypothesis, $f$ (bounded by a 4-cycle) is not adjacent to $^{\leq}7$-cycles. It follows that $G \in \mathcal{G}$. Finally observe that $c(G') < c(G)$ ($G'$ contains one 4-cycle less).

Let $f_0'$ be the outer face of $G'$. We now show that the precoloring $\phi'$ of $G'[V(f_0')]$ in $G'$ induced by the precoloring of $G[V(f_0)]$ in $G$ is well defined, i.e. by the identification of $x_1$ and $x_3$ of $f$, we do not identify two precolored vertices having different colors or create an edge between two precolored vertices having the same color. We consider the different cases according $C_0 \cap C_f$:

(1) If all the vertices of $f$ are inner vertices, we are done.
(2) Assume that $|C_0 \cap C_f| = 1$. Assume that $C_0 \cap C_f = \{x_1\}$ (by renaming the vertices of $f$ is necessary). Observe that $x_3$ has no neighbor on $C_0$ (otherwise, there exists a separating $^{\leq}11$-cycle that separates $x_2$ or $x_4$, contradicting Claim 3). Hence, the identification of $x_1$ and $x_3$ does not damage the precoloring of $G'[V(f_0')]$.
(3) Assume that $|C_0 \cap C_f| = 2$. Observe that if $C_0 \cap C_f = \{x_1, x_3\}$ or $\{x_2, x_4\}$, then the cycles' adjacency conditions implies that $|f_0| \geq 12$, contradicting the hypothesis. Assume that $C_0 \cap C_f = \{x_1, x_4\}$. Observe now that $x_2$ and $x_3$ cannot have both one neighbor on $C_0$ different from $x_1$ and $x_4$ respectively (otherwise, by the cycles' adjacency conditions, $|f_0| > 11$). Hence if $x_3$ has no neighbor on $C_0$ different from $x_4$, the identification of $x_1$ and $x_3$ does not damage the precoloring of $G'[V(f_0')]$. Now, if $x_3$ has a neighbor on $C_0$, then $x_2$ has no neighbor on $C_0$ different from $x_1$. Hence instead of identifying $x_1$ and $x_3$, we identify $x_2$ and $x_4$ (that corresponds to rename the vertices of $f$) and we are done.
(4) Assume that $|C_0 \cap C_f| = 3$. By the cycles' adjacency conditions, $C_0 \cap C_f$ is a set of consecutive vertices on $C_0$. Assume that $C_0 \cap C_f = \{x_4, x_1, x_2\}$ (by renaming the vertices of $f$ if necessary). By the cycles' adjacency conditions, the vertex $x_3$ has no neighbor on $C_0$ different from $x_2$ and $x_4$. Hence, the identification of $x_1$ and $x_3$ does not damage the precoloring of $G'[V(f_0')]$.
(5) Assume that $|C_0 \cap C_f| = 4$. Since $G[V(f_0)]$ is chordless, this implies that $f_0 = f$, contradicting Claim 5.

It follows that, by minimality of $G$, the precoloring of $G'[V(f_0')]$ can be extended to the whole graph $G'$. This gives an extension of the precoloring of $G[V(f_0)]$ to $G$ (by assigning to $x_1$ and $x_3$ the color of $x'$), a contradiction that completes the proof. □

Using similar arguments one can prove:

**Claim 7.** *G contains no internal i-faces with $i = 5, 6$.*

**Sketch of the proof.** Assume that $G$ contains a face $f = x_1x_2x_3x_4x_5$ (resp. $f = x_1x_2x_3x_4x_5x_6$) and $C_f = b(f)$. We identify $x_1$ and $x_3$ (resp. $x_1$ and $x_4$) and let $x'$ be the obtained vertex. Let $G'$ be the graph obtained from $G$ after the identification. Firstly, using the cycles' adjacency conditions and Claim 3, one can prove that the identification does not create a $^{\leq}7$-cycle except $x'x_4x_5$ (resp. $x'x_2x_3$ and $x'x_5x_6$) and does not create an adjacency between two $^{\leq}7$-cycles. Moreover $c(G') < c(G)$ since $G'$ contains one 5-cycle less (resp. one 6-cycle less). Secondly, one can prove that the identification does not violate the precoloring $\phi'$ of $G'[V(f_0')]$ (where $f_0'$ is the outer face of $G'$) induced by the precoloring $\phi$ of $G[V(f_0)]$ by checking the different cases according to $C_0 \cap C_f$. If all the vertices of $f$ are inner vertices, we are done. By the cycles' adjacency conditions, $C_0 \cap C_f$ is a set of consecutive vertices on $C_0$. Now, if $|C_0 \cap C_f| \leq 3$, then we can rename the vertices of $f$ such that by the cycles' adjacency conditions and Claim 3, the identification of $x_1$ and $x_3$ (resp. $x_1$ and $x_4$) does not damage the $\phi'$. Consider $|f| = 5$. Now, if $|C_0 \cap C_f| = 4$ and suppose $C_0 \cap C_f = \{x_4, x_5, x_1, x_2\}$ (by renaming the vertices of $f$ if necessary), then by the cycles' adjacency conditions, the vertex $x_3$ has no neighbor on $C_0$ different from $x_2$ and $x_4$. Let $C'' = C_0[x_2, x_4] \cup x_4x_3x_2$. By Claim 2, $x_3$ has a neighbor in $int(C'')$ and $out(C'') \neq \emptyset$. It follows by Claim 3 that $|C''| \geq 12$ and so $|f_0| > 11$, a contradiction. Finally if $|C_0 \cap C_f| = 5$, then, since $G[V(f_0)]$ is chordless, this implies that $f_0 = f$, contradicting Claim 5. The same arguments work when $|f| = 6$. □

**Claim 8.** *G contains no internal 7-faces.*

**Proof.** Assume that it exists an internal 7-face. Let $f = x_1x_2x_3x_4x_5x_6x_7$ be such an internal 7-face (the $x_i$'s appear in the clockwise order on $f$) and $C_f = b(f)$.

**Observation 2.** *Let $u, v$ two vertices of $V(f)$. Let $P_{u,v}$ be an elementary path linking u and v such that $P_{u,v} \cap V(f) = \{u, v\}$ and $C_f(u, v) \in int(P_{u,v} \cup C_f[v, u])$. Let $P_{v,u}$ be a path linking u and v such that $P_{v,u} \cap V(f) = \{u, v\}$ and $C_f(v, u) \in int(P_{v,u} \cup C_f[u, v])$ (see Fig. 4). It may happen that $P_{u,v}$ or/and $P_{v,u}$ does not exist.*

*By the cycles' adjacency conditions or by Claim 3, three cases can occur according to the distance of u and v on $C_f$:*

- *If u and v are adjacent on $C_f$ (see Fig. 4.(1)), the paths $P_{u,v}$ and $P_{v,u}$ have at least 8 vertices since there is no 7-cycle adjacent to $^{\leq}7$-cycles.*
- *If u and v are at distance two on $C_f$ (see Fig. 4.(2)), the paths $P_{u,v}$ and $P_{v,u}$ have at least 8 vertices and 11 vertices, respectively; since otherwise $P_{u,v} \cup C_f[v, u]$ (or $C_f[u, v] \cup P_{v,u}$) is a separating $^{\leq}11$-cycle.*

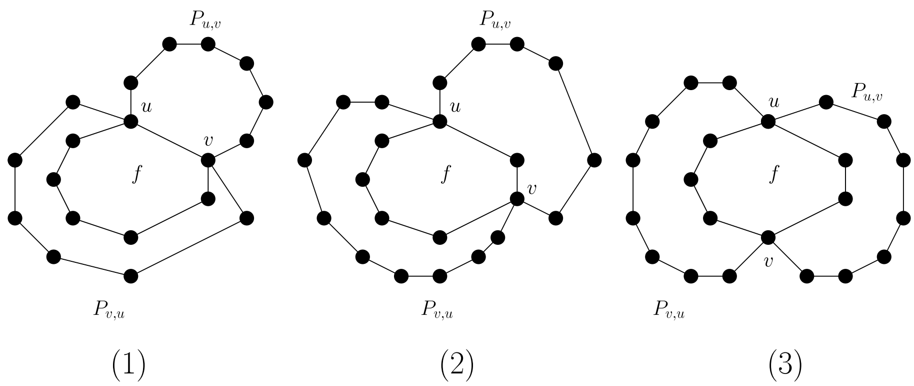


**Fig. 4.** The paths $P_{u,v}$ and $P_{v,u}$.

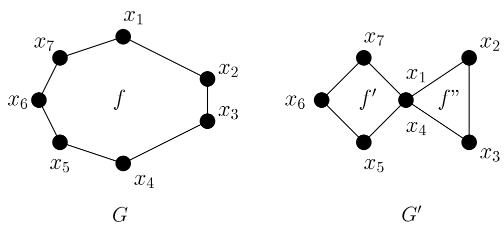


**Fig. 5.** The identification of $x_1$ with $x_4$.

- *If u and v are at distance three on $C_f$ (see Fig. 4.(3)), the paths $P_{u,v}$ and $P_{v,u}$ have at least* 9 *vertices and* 10 *vertices, respectively; since otherwise $P_{u,v} \cup C_f[v, u]$ (or $C_f[u, v] \cup P_{v,u}$) is a separating $^{\leq}11$-cycle.*

Let $G'$ be the graph obtained from $G$ by identifying $x_1$ with $x_4$, see Fig. 5. Let $x'$ be the identified vertex.

We will show that this identification does not create $^{\leq}7$-cycles, except $C_{f'} = x_1x_5x_6x_7$ and $C_{f''} = x_1x_2x_3$, which are a 4-cycle and a 3-cycle, respectively.

Suppose to the contrary that $C^*$ is a $^{\leq}7$-cycle in $G'$ created by the identification of $x_1$ and $x_4$ in $G$, different from $C_{f'}$ and $C_{f''}$. By $l(x, y)$ denote the distance between the vertices $x$ and $y$ in $(V(G), E(G) \setminus E(f))$, where $E(f)$ denotes the set of edges of $f$. The cycle $C^*$ must go through at least two vertices of $x_1, \ldots, x_7$ (otherwise, its length cannot decrease by the identification). Suppose that $C^*$ goes through $x$ and $y$ two vertices of $C_f$. Note that $l(x, y)$ is well defined and $|C^*| \geq l(x, y) + 1$, unless $\{x, y\} = \{x_1, x_4\}$. Moreover by Observation 2, $l(x, y) \geq 7$. Thus $|C^*| \geq 8$ unless $\{x, y\} = \{x_1, x_4\}$ and $l(x, y) = 7$. But in this case, Observation 2 ensures that $l(x, y) \geq 8$, a contradiction.

Hence, such a cycle $C^*$ cannot exist. The identification does not create $^{\leq}7$-cycles; moreover, by the cycles' adjacency conditions, $f$ is not adjacent to $^{\leq}7$-cycles; so it is for $f'$ and $f''$. It follows that the identification does not create a $^{\leq}7$-cycle adjacent to a $^{\leq}7$-cycle. This implies that $G' \in \mathcal{G}$.

Let $f'_0$ be the outer face of $G'$. We now show that the precoloring $\phi'$ of $G'[V(f'_0)]$ in $G'$ induced by the precoloring of $G[V(f_0)]$ in $G$ is well defined, i.e. by the identification of $x_1$ and $x_4$ of $f$, we do not identify two precolored vertices having different colors or create an edge between two precolored vertices having the same color.

By the cycles' adjacency conditions, one can observe that if $C_0 \cap C_f \neq \emptyset$, then $C_0 \cap C_f$ is a set of consecutive vertices on $C_0$.

We consider the different cases according $C_0 \cap C_f$:

(1) If all the vertices of $f$ are inner vertices, we are done.
(2) Assume that $|C_0 \cap C_f| \leq 4$. Suppose that $|C_0 \cap C_f| = 1$ and $C_0 \cap C_f = \{x_1\}$ (by renaming the vertices of $f$ is necessary) (resp. $|C_0 \cap C_f| = 2$ with $C_0 \cap C_f = \{x_7, x_1\}$, $|C_0 \cap C_f| = 3$ with $C_0 \cap C_f = \{x_7, x_1, x_2\}$, $|C_0 \cap C_f| = 4$ with $C_0 \cap C_f = \{x_6, x_7, x_1, x_2\}$). Observe that $x_4$ has no neighbor on $C_0$ by Claim 3. Hence, the identification of $x_1$ and $x_4$ does not damage the precoloring of $G'[V(f'_0)]$.
(3) Assume that $|C_0 \cap C_f| = 5$. Assume that $C_0 \cap C_f = \{x_6, x_7, x_1, x_2, x_3\}$. Observe that, by the cycles' adjacency conditions and Claim 3, $x_4$ has no neighbor on $C_0$ different from $x_3$. Let $C'' = C_0[x_3, x_6] \cup x_6x_5x_4x_3$. By Claim 2, $x_4$ is of degree 3. Hence it has a neighbor in $int(C'')$ and $out(C'') \neq \emptyset$. By Claim 3, $|C''| \geq 12$. It follows that $|f_0| \geq 13$, a contradiction.
(4) Assume that $|C_0 \cap C_f| = 6$. Assume that $C_0 \cap C_f = \{x_5, x_6, x_7, x_1, x_2, x_3\}$. Similarly, $x_4$ has no neighbor on $C_0$ different from $x_3$ and $x_5$. Let $C'' = C_0[x_3, x_5] \cup x_5x_4x_3$. By Claim 2, $x_4$ is of degree 3. Hence it has a neighbor in $int(C'')$ and $out(C'') \neq \emptyset$. By Claim 3, $|C''| \geq 12$. It follows that $|f_0| \geq 15$, a contradiction.
(5) Assume that $|C_0 \cap C_f| = 7$. Since $G[V(f_0)]$ is chordless, this implies that $f_0 = f$, contradicting Claim 5.

Finally, observe that $c(G') = c(G)$ and $\sigma(G') < \sigma(G)$. Hence we can conclude that, by minimality of $G$, we can extend the precoloring $\phi'$ to the whole graph $G'$ which gives an extension of the precoloring $\phi$ to $G$: set $\forall x \in V(G)\setminus\{x_1, x_4\}$, $\phi(x) = \phi'(x)$ and $\phi(x_1) = \phi(x_4) = \phi'(x')$.

This contradiction completes the proof of Claim 8. □

Hence, the counterexample $G$ to Lemma 1 contains no faces of size 4 to 7 and $|f_0| = 3, 8, 9, 10, 11$. If $8 \leq |f_0| \leq 11$, we can apply Theorem 4 and extend the coloring of $G[V(f_0)]$ to the whole graph, a contradiction. Now if $|f_0| = 3$, then by Theorem 1, $G$ is 3-colorable and hence by permuting the colors, we can extend the coloring of $G[V(f_0)]$ to the whole graph, a contradiction.

This completes the proof of Lemma 1.

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